Int. J. Heat Mass Transfer. Vol. 16, pp. 505–512. Pergamon Press 1973. Printed in Great Britain

HEAT TRANSFER TO A DRAINING FILM

J. ISENBERG and C. GUTFINGER
Laboratory for Coating Technology, Department of Mechanical Engineering, Technion—Israel Institute of Technology, Haifa, Israel

(*Received* 22 *December* 1971 *and in revised form* 26 *May* 1972)

Abstract—Heat transfer to an attached falling film has been investigated. This problem is unique because both the film thickness (the hydrodynamics) and the temperature profile change with time and position. The original problem which has three independent variables, two in space and one in time, is reduced by a similarity transformation to a two dimensional parabolic problem. The equation was solved numerically with a digital computer for five cases of boundary conditions.

NOMENCLATURE

- Bi, Biot number, hL/k,
- C, concentration;
- D, diffusivity of mass;
- g, gravitational constant;
- h, heat transfer coefficient;
- k, conductivity;
- L, characteristic length, $(\nu^2/gPr)^{1/3}$;
- Pr, Prandtl number, ν/α;
- q, heat flux per unit area;
- t, time;
- T, temperature;
- u, velocity component in x-direction;
- v, velocity component in y-direction;
- x, distance from leading edge;
- X, dimensionless distance from leading edge, x/L;
- y, transverse coordinate.

Greek symbols

- α, thermal diffusivity;
- δ. film thickness;
- η, dimensionless transverse coordinate, y/δ
- θ, dimensionless temperature, $(T - T_0)/(T_i - T_0)$;
- ν, kinematic viscosity;
- ξ, transformation variable, τ^2/X;
- τ, dimensionless time, $(g^2/\nu Pr)^{1/3}t$;
- ψ, boundary layer coordinate.

Subscripts

- i, temperature at $t = 0$;
- o, temperature at $x = 0$.

INTRODUCTION

HEAT transfer to fluids in unsteady motion is of interest in start-up operations of many industrial processes. A class of such unsteady flows is a liquid film of some initial thickness draining off a vertical plate. The present work considers the problem of diffusion or heat transfer associated with such a flow. Applications of this problem can be found in rinsing of objects removed from an electroplating solution (diffusion), cooling by a shower which forms a liquid film on the object (heat transfer). Moreover, this problem may be regarded as a prototype to the more complex problem of dip coating, a process involving diffusion of the solvent with a simultaneous property change within the film.

PROBLEM STATEMENT

(a) *Fluid mechanics*

Consider a thin film of incompressible liquid draining down a vertical wall as depicted in Fig. 1. The equation of motion for this film is

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} + g. \qquad (1)$$

For the case of thin films the left hand side of equation (1) representing inertial effects can be neglected for most practical purposes [1]. This results in an equation for creeping flow:

$$0 = v\frac{\partial^2 u}{\partial y^2} + g \quad (2)$$

where y is the transverse coordinate. This equation was integrated by Jeffreys [2] with a no-slip condition at the wall and no shear at the free surface of the film to yield the following velocity profile:

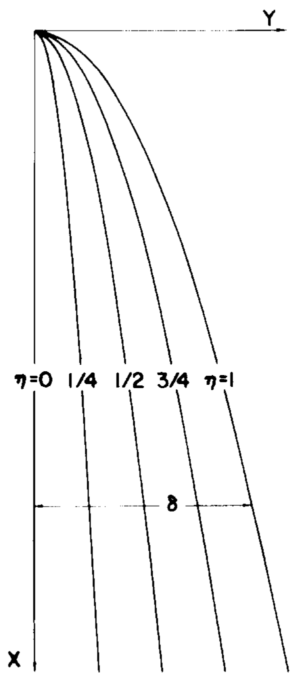

FIG. 1. Schematic of the coordinate system.

$$u = \frac{g}{v}\left(\delta y - \frac{y^2}{2}\right) \quad (3)$$

where δ denotes the film thickness. The variation of film thickness with time was introduced by means of a mass balance on a film element, Δx, resulting in

$$-\frac{\partial \delta}{\partial t} = \frac{\partial}{\partial x}\left[\int_0^\delta u \, \mathrm{d}y\right] \quad (4)$$

which upon integration yields the relationship for film thickness:

$$\delta = \sqrt{\left(\frac{vx}{gt}\right)}. \quad (5)$$

Equation (5) may now be substituted into equation (3) to yield a relationship for the velocity profile as a function of the three independent variables x, y and t:

$$u = \sqrt{\left(\frac{gx}{vt}\right)}y - \frac{g}{2v}y^2. \quad (6)$$

(b) *Heat transfer*

The governing equation for unsteady heat transfer within the film is*

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right). \quad (7)$$

Since the film is thin in comparison to its height, we may neglect vertical conduction using arguments similar to those of Ostrach [3], thus eliminating the term $\partial^2 T/\partial x^2$ in equation (7).

The velocity component in the y-direction, v, is small as compared to u, however the entire term $v\,\partial T/\partial y$ cannot be neglected as it is of the same magnitude as $u\,\partial T/\partial x$. The velocity component v can be evaluated by substituting equation (6) into the equation of continuity and integrating to yield

$$v = -\frac{1}{4}\sqrt{\left(\frac{g}{vtx}\right)}y^2. \quad (8)$$

The final form of the heat equation is

$$\frac{\partial T}{\partial t} + \left[\sqrt{\left(\frac{gx}{vt}\right)}y - \frac{g}{2v}y^2\right]\frac{\partial T}{\partial x} - \frac{1}{4}\sqrt{\left(\frac{g}{vtx}\right)}\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2}. \quad (9)$$

The equivalent mass diffusion problem may be formulated by replacing temperature T by concentration C and thermal diffusivity α by mass diffusivity D.

* An anonymous reviewer pointed out that the term $v(\partial T/\partial y)$ should be retained in the energy equation.

In this work, five cases of boundary conditions are treated:

Case 1:

$$\left.\begin{array}{lll} T(x,y,0) = T & x > 0 & 0 \leqslant u \leqslant \delta \\ T(0,y,t) = T_0 & t > 0 & 0 \leqslant y \leqslant \delta \\ T(x,0,t) = T_i & t > 0 & x > 0 \\ T(x,\delta,t) = T_0 & t > 0 & x > 0 \end{array}\right\} \quad (10)$$

Case 2:

$$\left.\begin{array}{lll} T(x,y,0) = T_i & x > 0 & 0 \leqslant y \leqslant \delta \\ T(0,y,t) = T_0 & t > 0 & 0 \leqslant y \leqslant \delta \\ \partial T/\partial y\,(x,0,t) = 0 & t > 0 & x > 0 \\ T(x,\delta,t) = T_0 & t > 0 & x > 0 \end{array}\right\} \quad (11)$$

Case 3:

$$\left.\begin{array}{lll} T(x,y,0) = T_i & x > 0 & 0 \leqslant y \leqslant \delta \\ T(0,y,t) = T_0 & t > 0 & 0 \leqslant y \leqslant \delta \\ T(x,0,t) = T_0 & t > 0 & x > 0 \\ T(x,\delta,t) = T_i & t > 0 & x > 0 \end{array}\right\} \quad (12)$$

Case 4:

$$\left.\begin{array}{lll} T(x,y,0) = T_i & x > 0 & 0 \leqslant y \leqslant \delta \\ T(0,y,t) = T_0 & t > 0 & 0 \leqslant y \leqslant \delta \\ T(x,0,t) = T_0 & t > 0 & x > 0 \\ \partial T/\partial y\,(x,\delta,t) = 0 & t > 0 & x > 0 \end{array}\right\} \quad (13)$$

Case 5:

$$\left.\begin{array}{lll} T(x,y,0) = T_i & x > 0 & 0 \leqslant y \leqslant \delta \\ T(0,y,t) = T_0 & t > 0 & 0 \leqslant y \leqslant \delta \\ T(x,0,t) = T_0 & t > 0 & x > 0 \\ -k\,\partial T/\partial y\,(x,\delta,t) & & \\ \quad = h(T - T_i) & t > 0 & x > 0 \end{array}\right\} \quad (14)$$

All cases treat the problem of heat transfer from a liquid film initially at temperature T_i. In Cases 1 and 2 the gas–liquid interface is kept at temperature T_0. The difference between the two cases lies in the boundary condition at the wall. In Case 1 the wall is maintained at the initial fluid temperature T_i, while in Case 2 the wall is insulated. In Cases 3 and 4 the wall is kept at temperature T_0. Here the difference between the two cases lies in the condition at the interface. In Case 3, the interface is maintained at the initial fluid temperature while in Case 4 it is insulated. Cases 3 and 4 are of interest because they provide an upper and lower bound for the actual situation where a convective boundary condition exists. This condition, given as Case 5, will be treated separately.

The energy equation will now be transformed into a dimensionless form using the following dimensionless variables:

$$\left.\begin{array}{ll} \theta \equiv \dfrac{T - T_0}{T_i - T_0} & \tau \equiv \left(\dfrac{g^2}{\nu Pr}\right)^{\frac{1}{3}} t \\ X \equiv \left(\dfrac{g}{\alpha\nu}\right)^{\frac{1}{3}} x = \left(\dfrac{gPr}{\nu^2}\right)^{\frac{1}{3}} x = \dfrac{x}{L}, & \eta \equiv \dfrac{y}{\delta} \end{array}\right\} \quad (15)$$

where L is a characteristic length and Pr denotes the Prandtl number. Rewriting equation (9) in terms of these dimensionless variables results in

$$\tau\frac{\partial\theta}{\partial\tau} + X\eta\left(1 - \frac{\eta}{2}\right)\frac{\partial\theta}{\partial X} = \frac{1}{2}\eta\left(\frac{3}{2}\eta - \frac{\eta^2}{2} - 1\right)\frac{\partial\theta}{\partial\eta} + \frac{\tau^2}{X}\frac{\partial^2\theta}{\partial\eta^2}. \quad (16)$$

The number of independent variables appearing in equation (16) will now be reduced from three to two by employing a similarity variable, $\xi = \tau^2/X$. This similarity transformation is admissible since it satisfies the boundary conditions.

The energy equation in the new coordinate system is

$$\tfrac{1}{2}(\eta^2 - 2\eta + 4)\frac{\partial\theta}{\partial\xi} = \frac{\partial^2\theta}{\partial\eta^2} - \frac{\eta}{4\xi}(\eta^2 - 3\eta + 2)\frac{\partial\theta}{\partial\eta}. \quad (17)$$

This equation was solved numerically for the five cases discussed above.

METHOD OF SOLUTION

Equation (17) is parabolic in ξ and although it is linear, it cannot be solved by known analytical methods. Therefore, a numerical technique was used to obtain the solution. Conventional finite difference methods such as Crank–Nicolson cannot be used for small ξ due to the appearance of ξ in the denominator of one of the terms. Therefore, the problem must be reformulated for this region. This is done by obtaining a starting solution, which is then extended in boundary layer coordinates until the entire region in η 'senses' the input boundary condition. We then revert to the regular $\eta - \xi$ coordinates for the main region. The method will be illustrated in detail using Case 1 defined by equation (10).

For Case 1 the boundary conditions of equation (10) in $\eta - \xi$ coordinates are

$$\left.\begin{array}{ll} \theta(\eta, 0) = 1 & 0 \leqslant \eta \leqslant 1 \\ \theta(0, \xi) = 1 & \xi > 0 \\ \theta(1, \xi) = 0 & \xi > 0. \end{array}\right\} \quad (18)$$

For the initial period (very small values of ξ) the region of temperature variation lies close to $\eta = 1$. In order to start the solution we examine an asymptotic form of equation (17) valid for very small ξ. We substitute $\eta = 1$ into equation (17) and obtain

$$\frac{3}{2}\frac{\partial\theta}{\partial\xi} = \frac{\partial^2\theta}{\partial\eta^2} - \frac{1-\eta}{4\xi}\frac{\partial\theta}{\partial\eta}. \quad (19)$$

We now employ a similarity variable

$$\psi = \frac{1-\eta}{\xi^{\frac{1}{2}}} \quad (20)$$

which transforms equation (19) from a partial to an ordinary differential equation:

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}\psi^2} + \psi\frac{\mathrm{d}\theta}{\mathrm{d}\psi} = 0 \quad (21)$$

with the boundary conditions

$$\theta = 0 \quad \text{at} \quad \psi = 0 \quad (22)$$

$$\theta = 1 \quad \text{at} \quad \psi \to \infty. \quad (23)$$

The solution of equation (21) is

$$\theta = \operatorname{erf}\left(\frac{\psi}{\sqrt{2}}\right) = \operatorname{erf}\left[\frac{1-\eta}{(2\xi)^{\frac{1}{2}}}\right]. \quad (24)$$

Having started the solution by the use of equation (24) we then employ the transformation of equation (20) as a boundary layer coordinate. In this system equation (17) becomes

$$\frac{\xi}{2}(3 + \psi^2\xi)\frac{\partial\theta}{\partial\xi} = \frac{\partial^2\theta}{\partial\psi^2} + \psi\frac{\partial\theta}{\partial\psi} \quad (25)$$

Equation (25) is solved numerically for increments in ξ. We return to the original $\eta - \xi$ system given by equation (17) when the region near $\eta = \theta$ starts to drop in temperature from its initial value.

Both equations (17) and (25) were solved numerically using the implicit Crank–Nicolson formulation.

The method of solution for Case 2 is the same as that outlined above. For Cases 3 and 4 a different transformation was used since the input boundary condition in η was on the opposite side. The transformation is

$$\psi \equiv \frac{\eta}{\xi^{\frac{1}{2}}} \quad (26)$$

which yields a starting solution given by

$$\theta = \operatorname{erf}\left(\frac{\psi}{2}\right) = \operatorname{erf}\left(\frac{\eta}{2\xi^{\frac{1}{2}}}\right). \quad (27)$$

RESULTS

The results of this work were obtained by using an IBM 370/165 computer.

Figures 3–5 show temperature profiles for the 4 cases outlined above. For compactness, the results are given in θ–η coordinates. Recall that η is defined as y/δ. Thus, lines of constant η appear in the physical plane as a family of parabolas as seen in Fig. 1. The second variable, ξ, has been defined as τ^2/X. Thus, each of the Figs. 2–5 can be interpreted as a history of the temperature profile at a constant X.

Comparing Cases 1 and 2 as depicted in Figs. 2 and 3, one notes that the constant wall temperature of Case 1 brings it faster to steady state than in the case of the insulated wall, Case 2. Conversely, a superposition of temperature profiles for these two cases shows that for short times, or large distances from the leading edge (small ξ), the two cases are identical. For intermediate values of ξ, say 0·2, the part of the profile close to the free surface is identical, indicating that they are insensitive to the boundary condition prevailing at the wall.

Similar observations can be made concerning

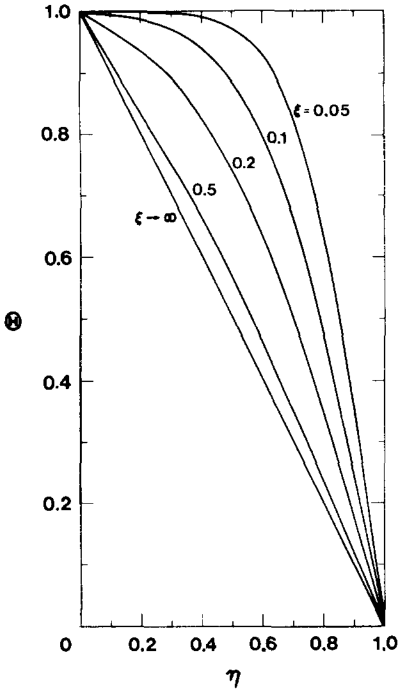

FIG. 2. Dimensionless temperature, θ, as a function of dimensionless transverse coordinate, η, for Case 1.

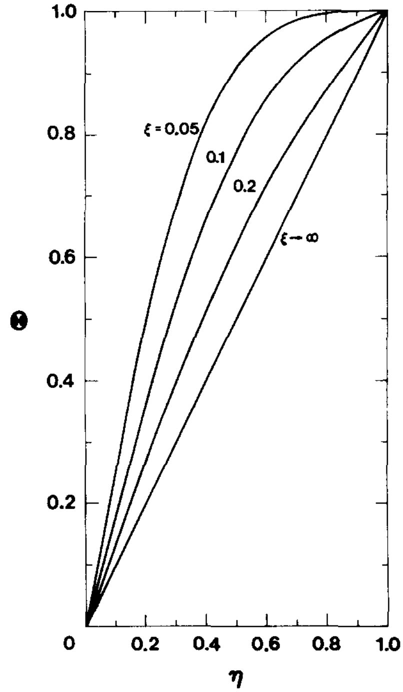

FIG. 4. Dimensionless temperature, θ, as a function of dimensionless transverse coordinate, θ, for Case 3.

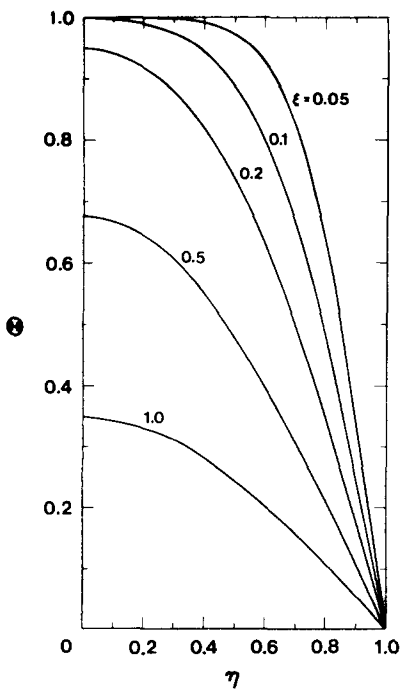

FIG. 3. Dimensionless temperature, θ, as a function of dimensionless transverse coordinate, η, for Case 2.

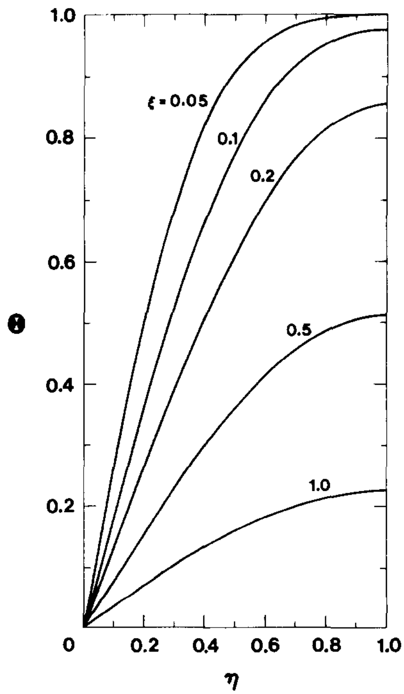

FIG. 5. Dimensionless temperature, θ, as a function of dimensionless transverse coordinate, η, for Case 4.

a comparison of Cases 3 and 4. However, it should be noted that in general, these two cases attain steady state faster than Cases 1 and 2. This is due to the fact that the velocity of the falling film is largest in the region of the free interface where its effect is most pronounced.

Cases 3 and 4 represent extreme cases for convective heat transfer at the free interface as given by

$$-k\frac{\partial T}{\partial y} = h(T - T_i) \tag{28}$$

where Cases 3 and 4 correspond respectively to $h \to \infty$ and $h = 0$. Therefore, they provide an upper and lower bound to equation (28). It is now of interest to examine the intermediate case for finite values of h.

CONVECTION AT THE FREE INTERFACE (CASE 5)

This problem differs from the previous ones only by the boundary condition at the free interface as given by equation (28).

Equation (28) can be recast in terms of the dimensionless variables θ and η to yield

$$-\frac{k}{\delta}\frac{\partial\theta}{\partial\eta} = h(\theta - 1). \tag{29}$$

The variable δ shown here is defined by equation (5). If rewritten in terms of τ and X, it becomes

$$\delta = \left(\frac{v^2}{gPr}\right)^{\frac{1}{3}}\left(\frac{X}{\tau}\right)^{\frac{1}{2}} = L\left(\frac{X}{\tau}\right)^{\frac{1}{2}}. \tag{30}$$

Now X may be eliminated from equation (30) by substituting $X = \tau^2/\xi$ to yield

$$\delta = \left(\frac{v^2}{gPr}\right)^{\frac{1}{3}}\left(\frac{\tau}{\xi}\right)^{\frac{1}{2}}. \tag{31}$$

Substituting (31) into equation (29) results in

$$-\frac{\partial\theta}{\partial\eta} = \frac{h}{k}\left[\left(\frac{v^2}{gPr}\right)^{\frac{1}{3}}\right]\left(\frac{\tau}{\xi}\right)^{\frac{1}{2}}(\theta - 1) \tag{32}$$

where the expression appearing in the square brackets is recognized as the characteristic length, defined in equation (15). It is convenient to define a Biot number given by

$$Bi \equiv \frac{h}{k}\left(\frac{v^2}{gPr}\right)^{\frac{1}{3}}. \tag{33}$$

With this Biot number, equation (32) takes on the simple form:

$$-\frac{\partial\theta}{\partial\eta} = Bi\left(\frac{\tau}{\xi}\right)^{\frac{1}{2}}(\theta - 1). \tag{34}$$

Equation (34) is the dimensionless form of the boundary condition given by equation (29). As seen in the convective case both the Biot number and the dimensionless time τ appear in the boundary condition. Thus, while in all previous cases the temperature profile was a function of η and ξ only, in the present case the temperature depends on η, ξ, τ and the parameter Bi. It should be noted that the field equation for this case still depends on two variables η and ξ, while it is through the boundary condition, that τ and Bi enter the problem. Thus, one may consider τ as a parameter rather than a variable for this case, and lump it together with Bi. The temperature variation is now specified as

$$\theta = \theta(\eta, \xi, Bi\sqrt{\tau}). \tag{35}$$

Figure 6 summarizes the computational results of temperature profiles for $Bi\sqrt{\tau}$ values ranging from 1 to 100. The results given in this figure are bounded by Fig. 5, the insulated air–liquid interface, which corresponds to $Bi\sqrt{\tau} = 0$, and Fig. 4, the constant temperature air–liquid interface corresponding to $Bi\sqrt{\tau} \to \infty$. The plots for $Bi\sqrt{\tau}$ values of 1 and 100 differ only slightly from the respective figures mentioned above. Thus the set of Figs. 4–6, encompasses the entire range of $Bi\sqrt{\tau}$ values. The curves in Fig. 6 are shown for various values of $Bi\sqrt{\tau}$ and ξ. It is noted that with decreasing ξ the curves for different values of $Bi\sqrt{\tau}$ tend to coincide. This indicates a weak dependence on $Bi\sqrt{\tau}$ in the region of small ξ.

HEAT TRANSFER DATA

Heat transfer results are presented in dimensionless form in Figs. 7 and 8. The local heat

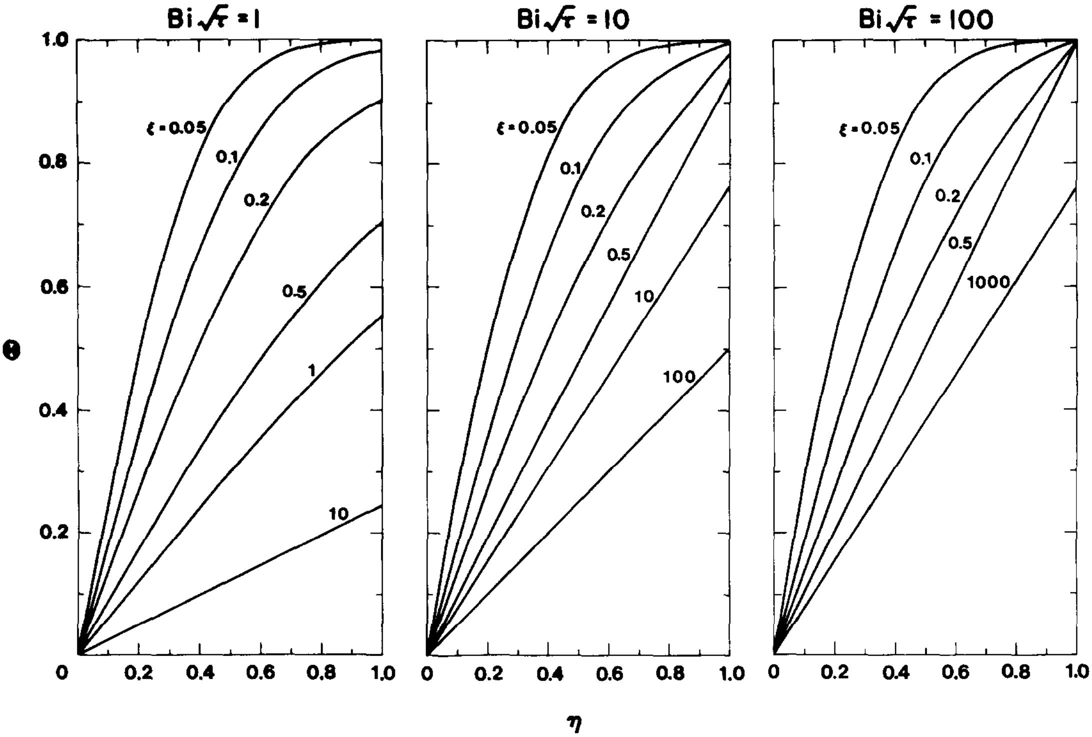

FIG. 6. Dimensionless temperature, θ, as a function of dimensionless transverse coordinate, η, for Case 5.

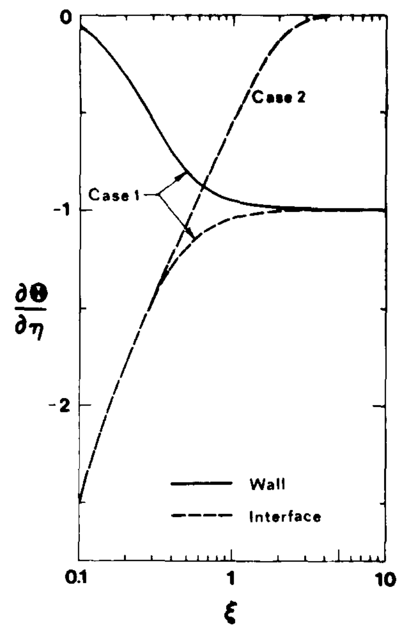

FIG. 7. Dimensionless heat transfer rate, $\partial\theta/\partial\eta$, as a function of ξ for Cases 1 and 2.

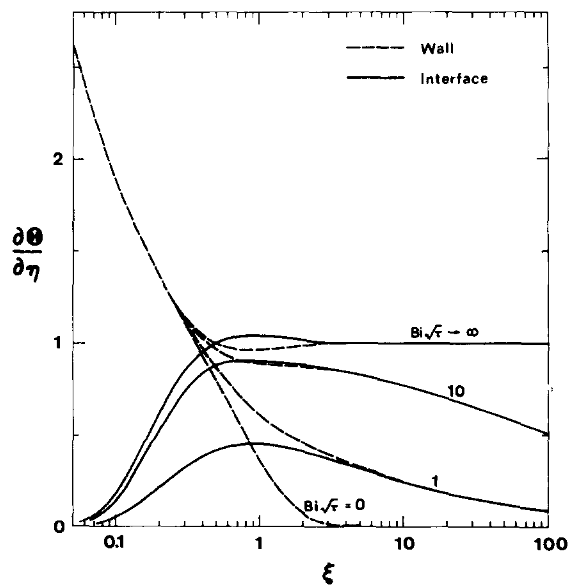

FIG. 8. Dimensionless heat transfer rate $\partial\theta/\partial\eta$, as a function of ξ for Cases 3, 4 and 5.

transfer rate per unit area is

$$q = -k\frac{\partial T}{\partial y} \tag{36}$$

which in dimensionless form reduces to

$$\frac{q}{(T_0 - T_i)\,k/\delta} = \frac{\partial \theta}{\partial \eta}. \tag{37}$$

Thus the local temperature gradient at the interfaces is a measure of the local heat transfer rates.

The dimensionless heat transfer rates are plotted on semi-logarithmic coordinates which were chosen in order to emphasize the asymptotic behaviour of heat transfer rates at large values of ξ. Two heat transfer rates are presented for each case: at the wall and at the air–liquid interface. The two rates approach each other for large ξ indicating steady state. For the special case when one of the boundaries is insulated, the heat flux at the opposite boundary approaches zero for large ξ. The foregoing remarks are illustrated in Fig. 7, which presents heat transfer rates for Cases 1 and 2.

Heat transfer rates for Cases 3, 4 and 5 are shown in Fig. 8, Here Cases 3 and 4 constitute upper and lower bounds for intermediate values of $Bi\sqrt{\tau}$. Case 3 represents the limiting case for $Bi\sqrt{\tau} \rightarrow \infty$ while Case 4 represents $Bi\sqrt{\tau} = 0$. As shown in Fig. 8, at $Bi\sqrt{\tau} = 0$ the dimensionless heat transfer rate approaches a value of zero, while that for the case of $Bi\sqrt{\tau} \rightarrow \infty$ converges to a value of 1. All intermediate cases of $Bi\sqrt{\tau}$ attain heat transfer rates between these limits of 0 and 1.

REFERENCES

1. C. GUTFINGER and J. A. TALLMADGE, Some remarks on the problem of drainage of fluids on vertical surfaces, *A.I.Ch.E. Jl* **10**, 774 (1964).
2. H. JEFFREYS, Draining of a vertical plate, *Proc. Camb. Phil. Soc.* **26**, 204 (1930).
3. S. OSTRACH, Role of analysis in the solution of complex physical problems, Third International Heat Transfer Conference, Chicago, Illinois (August, 1966).

TRANSFERT THERMIQUE A UN FILM RUISSELANT

Résumé—On a étudié le transfert thermique à un film tombant. Ce problème est particulier puisque à la fois l'épaisseur du film (l'hydrodynamique) et le profil de température changent en fonction du temps et de la position. Le problème qui a trois variables indépendantes, deux d'espace et une de temps, est réduit par une transformation de similitude à un problême parabolique à deux dimensions. L'équation a été numériquement résolue à l'aide d'un calculateur digital pour cinq cas de conditions aux limites.

WÄRMEÜBERGANG AN EINEM ABLAUFENDEN FILM

Zusammenfassung—An einem benetzenden fallenden Film wurde der Wärmeübergang untersucht. Dieses Problem ist eindeutig, da sowohl die Filmdicke (die Hydrodynamik) als auch das Temperaturprofil sich mit Zeit und Ort ändern. Das ursprüngliche Problem, das drei unabhängige Variable besitzt, zwei räumliche und eine zeitliche Veränderliche, wird mit Hilfe einer Ähnlichkeitstransformation reduziert auf ein zweidimensionales parabolisches Problem. Die Gleichung wurde numerisch mit einem Digitalrechner gelöst und zwar für fünf verschiedene Randbedingungen.

ПЕРЕНОС ТЕПЛА К ПАДАЮЩЕЙ ПЛЕНКЕ

Аннотация—Исследовался перенос тепла к падающей пленке. Эта задача уникальна, потому что, как толщина пленки (гидродинамика), так и профиль температур изменяются в пространстве и времени. Исходная задача, включающая три независимых переменных, две величины, изменяющиеся в пространстве, и одну, изменяющуюся во времени, сводится с помощью автомодельных преобразований к двумерной параболической задаче. Уравнение решается численно на цифровой вычислительной машине для пяти случаев граничных условий.